FIVE-DIMENSIONAL NILPOTENT EVOLUTION ALGEBRAS

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Abstract. The paper is devoted to give a complete classification of five-dimension nilpotent evolution algebras over an algebraically closed field. We obtained a list of 27 isolated non-isomorphic nilpotent evolution algebras and 2 families of non-isomorphic algebras depending on one parameter.

1. Introduction

Evolution algebras were introduced in 2006 by Tian and Vojtechovsky [3] motivated by the evolution laws of genetics. In this sense, the multiplication in an evolution algebra expresses self-reproduction of non-Mendelian genetics. Evolution algebras are not defined by identities, and hence they do not form a variety of non-associative algebras; they are commutative, but not in general power-associative, and possess some distinguishing properties that lead to many interesting mathematical results.

The classification, up to isomorphism, of any class of algebras is a fundamental and very difficult problem. In [2] the authors classified nilpotent evolution algebras of dimension up to four over an algebraically closed field. The aim of this paper is to give a full classification of nilpotent evolution algebras of dimension up to five over an algebraically closed field $\mathbb{F}$.

The paper is organized as follows. In Section 2 we describe the method that we use to classify nilpotent evolution algebras, which also appeared in [2]. In Section 3 we list, up to isomorphism, all nilpotent evolution algebras of dimension up to four. In Section 4 the classification of five-dimensional nilpotent evolution algebras is given.

2. The method

Definition 2.1. (See [4].) *An* evolution algebra *is an algebra $\mathcal{E}$ containing a basis (as a vector space) $B = \{e_1, \ldots, e_n\}$ such that $e_ie_j = 0$ for any $1 \leq i < j \leq n$. A basis with this property is called a* natural basis*.*

Definition 2.2. *An* ideal $\mathcal{I}$ *of an evolution algebra $\mathcal{E}$ is an evolution algebra satisfying $\mathcal{E}\mathcal{I} \subseteq \mathcal{I}$.*

Given an evolution algebra $\mathcal{E}$, consider its *annihilator*

$$ann\left(\mathcal{E}\right) := \left\{x \in \mathcal{E} : x\mathcal{E} = 0\right\}.$$

Lemma 2.3. (See [1].) *Let $B = \{e_1, \ldots, e_n\}$ be a natural basis of an evolution algebra $\mathcal{E}$. Then*

$$ann\left(\mathcal{E}\right) = \operatorname{span}\left\{e_i : e_i^2 = 0\right\}.$$

Given an evolution algebra $\mathcal{E}$, we introduce the following sequence of subspaces:

$$\mathcal{E}^{\langle 1\rangle} = \mathcal{E}, \quad \mathcal{E}^{\langle k+1\rangle} = \mathcal{E}^{\langle k\rangle}\mathcal{E}.$$

Definition 2.4. *An algebra $\mathcal{E}$ is called* nilpotent *if there exists $n \in \mathbb{N}$ such that $\mathcal{E}^{\langle n\rangle} = 0$, and the minimal such number is called the* index of nilpotency*.*

Let $\mathcal{E}$ be an evolution algebra with a natural basis $B_{\mathcal{E}} = \{e_1, \ldots, e_m\}$ and V be a vector space with a basis $B_V = \{e_{m+1}, \ldots, e_n\}$. Let $\mathcal{Z}\left(\mathcal{E} \times \mathcal{E}, V\right)$ be the set of all bilinear maps $\theta : \mathcal{E} \times \mathcal{E} \longrightarrow V$ such that $\theta\left(e_i, e_j\right) = 0$ for all $i \neq j$. Set $\mathcal{E}_\theta = \mathcal{E} \oplus V$, and then define a multiplication on $\mathcal{E}_\theta$ by $e_i \star e_j = e_ie_j \mid_{\mathcal{E}} + \theta\left(e_i, e_j\right)$ if $1 \leq i, j \leq m$, and otherwise $e_i \star e_j = 0$. Then $\mathcal{E}_\theta$ is an evolution algebra with a natural basis $B_{\mathcal{E}_\theta} = \{e_1, \ldots, e_n\}$. The evolution algebra $\mathcal{E}_\theta$ is called a $(n-m)$*-dimensional annihilator extension* of $\mathcal{E}$ by V.

Let $\theta : \mathcal{E} \times \mathcal{E} \longrightarrow V$ be a symmetric bilinear map. Then the set $\theta^\perp = \{x \in \mathcal{E} : \theta(x, \mathcal{E}) = 0\}$ is called the *radical* of θ.

Lemma 2.5. (See [2, Lemma 11].) *Let $\theta \in \mathcal{Z}\left(\mathcal{E} \times \mathcal{E}, V\right)$. Then $ann(\mathcal{E}_\theta) = \Big(\theta^\perp \cap ann\left(\mathcal{E}\right)\Big) \oplus V$.*

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So, if V is a non-trivial vector space then the annihilator of $\mathcal{E}_\theta$ will be non-trivial.

Theorem 2.6. *Let $\mathcal{E}$ be an evolution algebra with $ann(\mathcal{E}) \neq 0$. Then there exist, up to isomorphism, a unique evolution algebra E, and a $\theta \in \mathcal{Z}(E \times E, V)$ with $\theta^\perp \cap ann(E) = 0$ such that $\mathcal{E} \cong E_\theta$ and $\mathcal{E}/ann(\mathcal{E}) \cong E$.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a natural basis of $\mathcal{E}$ such that $\{e_{m+1}, \ldots, e_n\}$ be a basis of $ann(\mathcal{E})$. Set $V = ann(\mathcal{E})$, and let E be a complement of V in $\mathcal{E}$ (i.e., $\mathcal{E} = E \oplus V$). Let $P : \mathcal{E} \longrightarrow E$ be the projection of $\mathcal{E}$ onto E (i.e., $P(e_i) = e_i$ if $1 \leq i \leq m$, and otherwise $P(e_i) = 0$). Define a multiplication on E by $e_i *_E e_j = P(e_i e_j)$ for $1 \leq i, j \leq m$. Then E is an evolution algebra. Moreover, $P(e_i e_j) = P(e_i) *_E P(e_j)$ for $1 \leq i, j \leq n$. Hence P is a homomorphism of evolution algebras. So, $\mathcal{E}/V \cong E$. Define a bilinear map $\theta : E \times E \longrightarrow V$ by $\theta(e_i, e_j) = e_i e_j - P(e_i e_j)$ for $1 \leq i, j \leq m$. Then $\theta \in \mathcal{Z}(E \times E, V)$ and $\theta^\perp \cap ann(E) = 0$. Therefore, E_θ is an evolution algebra. Further, $e_i \star_{E_\theta} e_j = e_i *_E e_j + \theta(e_i, e_j) = e_i e_j$ for all $1 \leq i, j \leq m$. So, E_θ is the same evolution algebra as $\mathcal{E}$. □

Now, for a linear map $f \in Hom(\mathcal{E}, V)$, if we define $\delta f : \mathcal{E} \times \mathcal{E} \longrightarrow V$ by $\delta f(e_i, e_j) = f(e_i e_j)$, then $\delta f \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$. Let $\mathcal{B}(\mathcal{E} \times \mathcal{E}, V) = \{\delta f : f \in Hom(\mathcal{E}, V)\}$. Then $\mathcal{B}(\mathcal{E} \times \mathcal{E}, V)$ is a subspace of $\mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$. We define $\mathcal{H}(\mathcal{E} \times \mathcal{E}, V)$ to be the quotient space $\mathcal{Z}(\mathcal{E} \times \mathcal{E}, V) / \mathcal{B}(\mathcal{E} \times \mathcal{E}, V)$. The equivalence class of $\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ is denoted $[\theta] \in \mathcal{H}(\mathcal{E} \times \mathcal{E}, V)$.

Lemma 2.7. *Let $\theta, \vartheta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ such that $[\theta] = [\vartheta]$. Then*

(1) *$\theta^\perp \cap ann(\mathcal{E}) = \vartheta^\perp \cap ann(\mathcal{E})$ or, equivalently, $ann(\mathcal{E}_\theta) = ann(\mathcal{E}_\vartheta)$.*
(2) *$\mathcal{E}_\theta \cong \mathcal{E}_\vartheta$.*

Proof. (1) Since $[\theta] = [\vartheta]$, $\vartheta = \theta + \delta f$ for some $f \in Hom(\mathcal{E}, V)$. So $\vartheta(x, y) = \theta(x, y) + f(xy)$ for all $x, y \in \mathcal{E}$. Hence $\theta(x, y) = xy = 0$ if and only if $\vartheta(x, y) = xy = 0$. Therefore $\theta^\perp \cap ann(\mathcal{E}) = \vartheta^\perp \cap ann(\mathcal{E})$. Then, by Lemma 2.5, $ann(\mathcal{E}_\theta) = ann(\mathcal{E}_\vartheta)$.
(2) See [2, Lemma 5].

□

Let $\mathcal{E}$ be an evolution algebra with a natural basis $B_\mathcal{E} = \{e_1, \ldots, e_m\}$ and V be a vector space with a basis $B_V = \{e_{m+1}, \ldots, e_n\}$. Then a $\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ can be uniquely written as $\theta(e_i, e_i) = \sum_{j=1}^{n-m} \theta_j(e_i, e_i) e_{m+j}$, where $\theta_j \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$. Moreover, $\theta \in \mathcal{B}(\mathcal{E} \times \mathcal{E}, V)$ if and only if all $\theta_j \in \mathcal{B}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$. Furthermore, $\theta^\perp = \theta_1^\perp \cap \ldots \cap \theta_{n-m}^\perp$.

Lemma 2.8. (See [2, Lemma 6].) *Let $\mathcal{E}$ be an n-dimensional nilpotent evolution algebra. Let $e_1, \ldots, e_m$ be a basis of $\mathcal{E}^{\langle 2 \rangle}$. Then $\dim \mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) = n - m$ and $\mathcal{B}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) = \langle \delta e_1^*, \ldots, \delta e_m^* \rangle$ where $e_i^*(e_j) = \delta_{ij}$ and δ_{ij} is the* Kronecker delta.

Let $\mathcal{E}$ be an evolution algebra with a natural basis $B_\mathcal{E} = \{e_1, \ldots, e_m\}$. We define δ_{e_i, e_j} to be the symmetric bilinear form $\delta_{e_i, e_j} : \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{F}$ with $\delta_{e_i, e_j}(e_l, e_m) = 1$ if $\{i, j\} = \{l, m\}$, and otherwise $\delta_{e_i, e_j}(e_l, e_m) = 0$. Denote the space of all symmetric bilinear forms on $\mathcal{E}$ by $Sym(\mathcal{E})$. Then $Sym(\mathcal{E}) = \left\langle \delta_{e_i, e_j} : 1 \leq i \leq j \leq m \right\rangle$ while $\mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) = \langle \delta_{e_i, e_i} : 1 = 1, 2, \ldots, m \rangle$.

Example 2.9. *Let $\mathcal{E} : e_1^2 = e_2$ be a 2-dimensional nilpotent evolution algebra. Then $\mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) = \langle \delta_{e_1, e_1}, \delta_{e_2, e_2} \rangle$. By Lemma 2.8, $\mathcal{B}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) = \langle \delta e_1^* \rangle$. Since $\mathcal{B}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$ is a subspace of $\mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$, $\delta e_2^* = \alpha \delta_{e_1, e_1} + \beta \delta_{e_2, e_2}$. Then $\alpha = \delta e_2^*(e_1, e_1) = e_1^*\left(e_1^2\right) = 1$ and $\beta = \delta e_2^*(e_2, e_2) = e_1^*\left(e_2^2\right) = 0$. So, $\mathcal{B}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) = \langle \delta_{e_1, e_1} \rangle$ and therefore $\mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) = \langle [\delta_{e_2, e_2}] \rangle$.*

Let $Aut(\mathcal{E})$ be the automorphism group of the algebra $\mathcal{E}$. Let $\phi \in Aut(\mathcal{E})$. For $\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ defining $\phi\theta(e_i, e_j) = \theta(\phi(e_i), \phi(e_j))$ then $\phi\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ if and only if $\theta(\phi(e_i), \phi(e_j)) = 0$ for any $i \neq j$. Consider the matrix representation and assume that $\theta = \sum_{i=1}^{m} c_{ii} \delta_{e_i, e_i} \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$. Then θ can be represented by an $m \times m$ diagonal matrix $\left(c_{ii}\right)$, and then $\phi\theta = \phi^t \left(c_{ii}\right) \phi$. So, $\phi\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ if and only if the matrix represents $\phi\theta$ is diagonal. Define a subgroup $\mathcal{S}_\theta(\mathcal{E})$ of $Aut(\mathcal{E})$ by

$$\mathcal{S}_\theta(\mathcal{E}) = \{\phi \in Aut(\mathcal{E}) : \theta(\phi(e_i), \phi(e_j)) = 0 \quad \forall i \neq j\}.$$

Then, for each $\phi \in \mathcal{S}_\theta(\mathcal{E})$, $\phi\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ and therefore $\mathcal{E}_{\phi\theta}$ is an evolution algebra. Further, $\phi\theta \in \mathcal{B}(\mathcal{E} \times \mathcal{E}, V)$ if and only if $\theta \in \mathcal{B}(\mathcal{E} \times \mathcal{E}, V)$ and hence $[\phi\theta]$ is a well-defined element of $\mathcal{H}(\mathcal{E} \times \mathcal{E}, V)$.

Lemma 2.10. *Let $\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ and $\phi \in \mathcal{S}_\theta(\mathcal{E})$. Then $\mathcal{E}_{\phi\theta} \cong \mathcal{E}_\theta$.*

Proof. Define a linear map $\sigma : \mathcal{E}_{\phi\theta} \longrightarrow \mathcal{E}_{\theta}$ by $\sigma(x+v) = \phi(x) + v$ for $x \in \mathcal{E}$ and $v \in V$. Then σ is a bijective map. Moreover,

$$\begin{aligned} \sigma((x+v)(y+w)) &= \sigma(xy + \phi\theta(x,y)) \\ &= \phi(xy) + \phi\theta(x,y) \\ &= \phi(x)\phi(y) + \theta(\phi(x), \phi(y))) \\ &= \sigma(x+v)\sigma(y+w) \end{aligned}$$

for any $x, y \in \mathcal{E}$ and $v, w \in V$. Hence σ is an isomorphism. □

Let $GL(V)$ be the set of bijective linear maps $V \longrightarrow V$. Let $\psi \in GL(V)$. For $\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ define $\psi\theta(e_i, e_j) = \psi(\theta(e_i, e_j))$. Then $\psi\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$. So, $GL(V)$ acts on $\mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$. Also, $\theta \in \mathcal{B}(\mathcal{E} \times \mathcal{E}, V)$ if and only if $\psi\theta \in \mathcal{B}(\mathcal{E} \times \mathcal{E}, V)$. Hence $GL(V)$ acts on $\mathcal{H}(\mathcal{E} \times \mathcal{E}, V)$.

Lemma 2.11. *Let $\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ and $\psi \in GL(V)$. Then $\mathcal{E}_{\theta} \cong \mathcal{E}_{\psi\theta}$.*

Proof. Define a linear map $\sigma : \mathcal{E}_{\theta} \longrightarrow \mathcal{E}_{\psi\theta}$ by $\sigma(x+v) = x + \psi(v)$ for $x \in \mathcal{E}$ and $v \in V$. Then σ is a bijective map. Moreover,

$$\sigma((x+v)(y+w)) = \sigma(xy + \theta(x,y)) = xy + \psi(\theta(x,y)) = \sigma(x+v)\sigma(y+w)$$

for any $x, y \in \mathcal{E}$ and $v, w \in V$. Hence σ is an isomorphism. □

Lemma 2.12. *Let $\theta, \vartheta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$. If there exist a map $\phi \in Aut(\mathcal{E})$ and a map $\psi \in GL(V)$ such that $[\phi\theta] = [\psi\vartheta]$, then $\mathcal{E}_{\theta} \cong \mathcal{E}_{\vartheta}$.*

Proof. Let $\phi \in Aut(\mathcal{E})$ and $\psi \in GL(V)$ such that $[\phi\theta] = [\psi\vartheta]$ (i.e., $\phi \in \mathcal{S}_{\theta}(\mathcal{E})$). Then, by Lemma 2.7, $\mathcal{E}_{\phi\theta} \cong \mathcal{E}_{\psi\vartheta}$. Moreover, by Lemma 2.10 and Lemma 2.11, $\mathcal{E}_{\theta} \cong \mathcal{E}_{\phi\theta} \cong \mathcal{E}_{\psi\vartheta} \cong \mathcal{E}_{\vartheta}$. □

Lemma 2.13. (See [2, Lemma 14].) *Let $\mathcal{E}$ be an evolution algebra with a natural basis $B_{\mathcal{E}} = \{e_1, \ldots, e_m\}$, V be a vector space with a basis $B_V = \{e_{m+1}, \ldots, e_n\}$, and $\theta, \vartheta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ with $\theta^{\perp} \cap ann(\mathcal{E}) = \vartheta^{\perp} \cap ann(\mathcal{E}) = 0$. Suppose that $\theta(e_i, e_i) = \sum_{j=1}^{n-m} \theta_j(e_i, e_i) e_{m+j}$ and $\vartheta(e_i, e_i) = \sum_{j=1}^{n-m} \vartheta_j(e_i, e_i) e_{m+j}$. Then $\mathcal{E}_{\theta} \cong \mathcal{E}_{\vartheta}$ if and only if there is a $\phi \in Aut(\mathcal{E})$ (more precisely, $\phi \in \mathcal{S}_{\vartheta}(\mathcal{E})$) such that the $[\phi\vartheta_i]$ span the same subspace of $\mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$ as the $[\theta_i]$.*

Definition 2.14. *Let $\mathcal{E}$ be an evolution algebra with a natural basis $B_{\mathcal{E}} = \{e_1, \ldots, e_m\}$. If $\mathcal{E} = \langle e_1, .., e_{r-1}, e_{r+1}, .., e_m\rangle \oplus \mathbb{F}e_r$ is the direct sum of two nilpotent ideals, then $\mathbb{F}e_r$ is called an* annihilator component *of $\mathcal{E}$.*

Lemma 2.15. (See [2, Lemma 16].) *Let $\mathcal{E}$ be an evolution algebra with a natural basis $B_{\mathcal{E}} = \{e_1, \ldots, e_m\}$, V be a vector space with a basis $B_V = \{e_{m+1}, \ldots, e_n\}$, and $\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$. Suppose that $\theta(e_i, e_i) = \sum_{j=1}^{n-m} \theta_j(e_i, e_i) e_{m+j}$ and $\theta^{\perp} \cap ann(\mathcal{E}) = 0$. Then $\mathcal{E}_{\theta}$ has an annihilator component if and only if $[\theta_1], [\theta_2], \ldots, [\theta_s]$ are linearly dependent in $\mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$.*

Lemma 2.16. *Let $\phi \in Aut(\mathcal{E})$ and $\theta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$. Then $\dim(\phi\theta)^{\perp} = \dim \theta^{\perp}$.*

Proof. Since $\phi\theta(x, \mathcal{E}) = \theta(\phi(x), \mathcal{E})$, $x \in (\phi\theta)^{\perp}$ if and only if $\phi(x) \in \theta^{\perp}$. Define a linear map $\sigma : (\phi\theta)^{\perp} \longrightarrow \theta^{\perp}$ by $\sigma(x) = \phi(x)$. Then σ is a bijective map. So, $\dim(\phi\theta)^{\perp} = \dim \theta^{\perp}$. □

For $\theta_1, \theta_2, \ldots, \theta_{n-m} \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$, let $\Psi(\theta_1, \theta_2, \ldots, \theta_{n-m}) = (m_1, m_2, \ldots, m_{n-m})$ be the ordered descending sequence of $\dim \theta_1^{\perp}, \dim \theta_2^{\perp}, ..., \dim \theta_{n-m}^{\perp}$.

Corollary 2.17. *Let $\phi \in Aut(\mathcal{E})$ and $\theta_1, \ldots, \theta_{n-m} \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$. Then $\Psi(\theta_1, \ldots, \theta_{n-m}) = \Psi(\phi\theta_1, \ldots, \phi\theta_{n-m})$.*

Let $[\theta_1], \ldots, [\theta_{n-m}] \in \mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$. We define $\Psi([\theta_1], \ldots, [\theta_{n-m}])$ to be the least upper bound for the lexicographic order of the set

$$\{\Psi(\vartheta_1, \ldots, \vartheta_{n-m}) : \vartheta_1, \ldots, \vartheta_{n-m} \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) \text{ and } \langle [\theta_1], ..., [\theta_{n-m}]\rangle = \langle [\vartheta_1], ..., [\vartheta_{n-m}]\rangle\}.$$

Lemma 2.18. *Let $\mathcal{E}$ be an evolution algebra with a natural basis $B_{\mathcal{E}} = \{e_1, \ldots, e_m\}$, V be a vector space with a basis $B_V = \{e_{m+1}, \ldots, e_n\}$, and $\theta, \vartheta \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, V)$ with $\theta^{\perp} \cap ann(\mathcal{E}) = \vartheta^{\perp} \cap ann(\mathcal{E}) = 0$. Suppose that $\theta(e_l, e_l) = \sum_{i=1}^{n-m} \theta_i(e_l, e_l) e_{m+i}$ and $\vartheta(e_l, e_l) = \sum_{i=1}^{n-m} \vartheta_i(e_l, e_l) e_{m+i}$. If $\mathcal{E}_{\theta} \cong \mathcal{E}_{\vartheta}$ then $\Psi(\langle [\theta_1], \ldots, [\theta_{n-m}]\rangle) = \Psi(\langle [\vartheta_1], \ldots, [\vartheta_{n-m}]\rangle)$.*

Proof. Let $\mathcal{E}_\theta \cong \mathcal{E}_\vartheta$. Then by Lemma 2.13, there exists $\phi \in Aut(\mathcal{E})$ such that $\langle [\phi\vartheta_1], \ldots, [\phi\vartheta_{n-m}] \rangle = \langle [\theta_1], \ldots, [\theta_{n-m}] \rangle$. Then $\Psi(\langle [\phi\vartheta_1], \ldots, [\phi\vartheta_{n-m}] \rangle) = \Psi(\langle [\theta_1], \ldots, [\theta_{n-m}] \rangle)$. Further, let $\Psi([\phi\vartheta_1], \ldots, [\phi\vartheta_{n-m}]) = \Psi(\eta_1, \ldots, \eta_{n-m})$ for some $\eta_1, \ldots, \eta_{n-m} \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$. Then $\langle [\phi\vartheta_1], \ldots, [\phi\vartheta_{n-m}] \rangle = \langle [\eta_1], \ldots, [\eta_{n-m}] \rangle$ and therefore $\langle [\vartheta_1], \ldots, [\vartheta_{n-m}] \rangle = \langle [\phi^{-1}\eta_1], \ldots, [\phi^{-1}\eta_{n-m}] \rangle$. By Corollary 2.17, $\Psi(\eta_1, \ldots, \eta_{n-m}) = \Psi(\phi^{-1}\eta_1, \ldots, \phi^{-1}\eta_{n-m})$ and hence $\Psi(\langle [\vartheta_1], \ldots, [\vartheta_{n-m}] \rangle) \geq \Psi(\eta_1, \ldots, \eta_{n-m})$. So, $\Psi(\langle [\vartheta_1], \ldots, [\vartheta_{n-m}] \rangle) \geq \Psi([\phi\vartheta_1], \ldots, [\phi\vartheta_{n-m}])$. On the other hand, let $\Psi(\langle [\vartheta_1], \ldots, [\vartheta_{n-m}] \rangle) = \Psi(\eta'_1, \ldots, \eta'_{n-m})$ for some $\eta'_1, \ldots, \eta'_{n-m} \in \mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$. Then $\langle [\vartheta_1], \ldots, [\vartheta_{n-m}] \rangle = \langle [\eta'_1], \ldots, [\eta'_{n-m}] \rangle$ and therefore $\langle [\phi\vartheta_1], \ldots, [\phi\vartheta_{n-m}] \rangle = \langle [\phi\eta'_1], \ldots, [\phi\eta'_{n-m}] \rangle$. So, by Corollary 2.17, $\Psi(\langle [\vartheta_1], \ldots, [\vartheta_{n-m}] \rangle) \leq \Psi(\langle [\phi\vartheta_1], \ldots, [\phi\vartheta_{n-m}] \rangle)$. This completes the proof. □

Now we carry out a procedure that takes as input a nilpotent evolution algebra $\mathcal{E}$ of dimension m with a natural basis $B_{\mathcal{E}} = \{e_1, \ldots, e_m\}$. It gives us all nilpotent evolution algebras $\tilde{\mathcal{E}}$ of dimension n with a natural basis $B_{\tilde{\mathcal{E}}} = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ such that $\tilde{\mathcal{E}}/ann\left(\tilde{\mathcal{E}}\right) \cong \mathcal{E}$ and $\tilde{\mathcal{E}}$ has no annihilator components. The procedure runs as follows:

1. Determine $\mathcal{Z}(\mathcal{E} \times \mathcal{E}, \mathbb{F}), \mathcal{B}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$ and $\mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$.
2. Let V be a vector space with a basis $B_V = \{e_{m+1}, \ldots, e_n\}$. Consider $[\theta] \in \mathcal{H}(\mathcal{E} \times \mathcal{E}, V)$ with
$$\theta(e_i, e_i) = \sum_{j=1}^{n-m} \theta_j(e_i, e_i)\, e_{m+j},$$
where the $[\theta_j] \in \mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$ are linearly independent, and $\theta^\perp \cap ann(\mathcal{E}) = 0$. Use the action of $\mathcal{S}_\theta(\mathcal{E})$ on $[\theta]$ by hand calculations to get a list of orbit representatives as small as possible.
3. For each θ found, construct $\mathcal{E}_\theta$. Get rid of isomorphic ones.

3. Nilpotent evolution algebras of dimension up to four

Here we list all nilpotent evolution algebras over an algebraically closed field $\mathbb{F}$ of dimension up to four. We denote the j-th algebra of dimension i with a natural basis $\{e_1, \ldots, e_i\}$ by $\mathcal{E}_{i,j}$, and among the natural basis of $\mathcal{E}_{i,j}$ the multiplication is specified by giving only the nonzero products. Throughout the paper, we describe the automorphism group, $Aut(\mathcal{E})$, of an evolution algebra $\mathcal{E}$ using column convention: the action of a $\phi \in Aut(\mathcal{E})$ on the i-th basis element of $\mathcal{E}$ is given by i-th column of the matrix of ϕ.

Theorem 3.1. *(See* [2]*.) Any nilpotent evolution algebra of dimension* ≤ 3 *over an algebraically closed field* $\mathbb{F}$ *is isomorphic to one of the following algebras:*

$\mathcal{E}$	*Multiplication Table*	$\mathcal{B}(\mathcal{E}\times\mathcal{E},\mathbb{F})$	$\mathcal{H}(\mathcal{E}\times\mathcal{E},\mathbb{F})$	$ann(\mathcal{E})$	$Aut(\mathcal{E})$
$\mathcal{E}_{1,1}$			$Sym(\mathcal{E}_{1,1})$	$\mathcal{E}_{1,1}$	$GL(\mathcal{E}_{1,1})$
$\mathcal{E}_{2,1}$			$Sym(\mathcal{E}_{2,1})$	$\mathcal{E}_{2,1}$	$GL(\mathcal{E}_{2,1})$
$\mathcal{E}_{2,2}$	$e_1^2 = e_2$	$\langle \delta_{e_1,e_1} \rangle$	$\langle [\delta_{e_2,e_2}] \rangle$	$\langle e_2 \rangle$	$\phi = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{11}^2 \end{bmatrix}$
$\mathcal{E}_{3,1}$			$Sym(\mathcal{E}_{3,1})$	$\mathcal{E}_{3,1}$	$GL(\mathcal{E}_{3,1})$
$\mathcal{E}_{3,2}$	$e_1^2 = e_2$	$\langle \delta_{e_1,e_1} \rangle$	$\langle [\delta_{e_2,e_2}], [\delta_{e_3,e_3}] \rangle$	$\langle e_2, e_3 \rangle$	$\phi = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11}^2 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}$
$\mathcal{E}_{3,3}$	$e_1^2 = e_3, e_2^2 = e_3$	$\langle \delta_{e_1,e_1} + \delta_{e_2,e_2} \rangle$	$\langle [\delta_{e_1,e_1}], [\delta_{e_3,e_3}] \rangle$	$\langle e_3 \rangle$	$\phi = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $a_{11}^2 + a_{21}^2 = a_{33}$, $a_{12}^2 + a_{22}^2 = a_{33}$, $a_{11}a_{12} + a_{21}a_{22} = 0$.
$\mathcal{E}_{3,4}$	$e_1^2 = e_2, e_2^2 = e_3$	$\langle \delta_{e_1,e_1}, \delta_{e_2,e_2} \rangle$	$\langle [\delta_{e_3,e_3}] \rangle$	$\langle e_3 \rangle$	$\phi = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{11}^2 & 0 \\ a_{31} & 0 & a_{11}^4 \end{bmatrix}$

TABLE 1. Nilpotent evolution algebras of dimension up to three.

Let us now classify 4-dimensional nilpotent evolution algebras. For this, we have the following subsections.

3.1. **Nilpotent evolution algebras with annihilator components.** By Definition 2.14, we get the algebras $\mathcal{E}_{4,i} = \mathcal{E}_{3,i} \oplus \mathcal{E}_{1,1}$, $i = 1, \ldots, 4$.

3.2. **1-dimensional annihilator extensions of $\mathcal{E}_{3,1}$.** Let $[\theta] = \alpha_1 [\delta_{e_1,e_1}] + \alpha_2 [\delta_{e_2,e_2}] + \alpha_3 [\delta_{e_3,e_3}] \in \mathcal{H}(\mathcal{E}_{3,1} \times \mathcal{E}_{3,1}, \mathbb{F})$ such that $\theta^{\perp} \cap ann(\mathcal{E}_{3,1}) = 0$. Then $\alpha_1\alpha_2\alpha_3 \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{-\frac{1}{2}} & 0 & 0 \\ 0 & \alpha_2^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha_3^{-\frac{1}{2}} \end{bmatrix}.$$

Then $[\phi\theta] = [\delta_{e_1,e_1}] + [\delta_{e_2,e_2}] + [\delta_{e_3,e_3}]$. So, we may assume $\theta = \delta_{e_1,e_1} + \delta_{e_2,e_2} + \delta_{e_3,e_3}$. Hence we get the algebra $\mathcal{E}_{4,5} : e_1^2 = e_2^2 = e_3^2 = e_4$.

3.3. **1-dimensional annihilator extensions of $\mathcal{E}_{3,2}$.** Let $[\theta] = \alpha_1 [\delta_{e_2,e_2}] + \alpha_2 [\delta_{e_3,e_3}] \in \mathcal{H}(\mathcal{E}_{3,2} \times \mathcal{E}_{3,2}, \mathbb{F})$ with $\alpha_1\alpha_2 \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{-\frac{1}{4}} & 0 & 0 \\ 0 & \alpha_1^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha_2^{-\frac{1}{2}} \end{bmatrix}.$$

Then $[\phi\theta] = [\delta_{e_2,e_2}] + [\delta_{e_3,e_3}]$. So, we may assume $\theta = \delta_{e_2,e_2} + \delta_{e_3,e_3}$. Hence we get the algebra $\mathcal{E}_{4,6} : e_1^2 = e_2, e_2^2 = e_3^2 = e_4$.

3.4. **1-dimensional annihilator extensions of $\mathcal{E}_{3,3}$.** Let $[\theta] = \alpha_1 [\delta_{e_1,e_1}] + \alpha_2 [\delta_{e_3,e_3}] \in \mathcal{H}(\mathcal{E}_{3,3} \times \mathcal{E}_{3,3}, \mathbb{F})$ with $\alpha_2 \neq 0$. If $\alpha_1 = 0$, then $[\theta] = \alpha_2 [\delta_{e_3,e_3}]$. Since $\alpha_2 [\delta_{e_3,e_3}]$ spans the same subspace as $[\delta_{e_3,e_3}]$, we may assume that $\theta = \delta_{e_3,e_3}$. Thus we get the algebra $\mathcal{E}_{4,7} : e_1^2 = e_3, e_2^2 = e_3, e_3^2 = e_4$. If $\alpha_1 \neq 0$, we choose ϕ to be the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{\frac{1}{2}}\alpha_2^{-\frac{1}{2}} & 0 & 0 \\ 0 & \alpha_1^{\frac{1}{2}}\alpha_2^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha_1\alpha_2^{-1} \end{bmatrix}.$$

Then $[\phi\theta] = \alpha_1^2\alpha_2^{-1} ([\delta_{e_2,e_2}] + [\delta_{e_3,e_3}])$. So, we may assume $\theta = \delta_{e_2,e_2} + \delta_{e_3,e_3}$. Hence we get the algebra $\mathcal{E}_{4,8} : e_1^2 = e_3 + e_4, e_2^2 = e_3, e_3^2 = e_4$. Further, $\mathcal{E}_{4,7}$ is not isomorphic to $\mathcal{E}_{4,8}$ since $\Psi(\langle [\delta_{e_3,e_3}] \rangle) = (2)$ while $\Psi(\langle [\delta_{e_2,e_2}] + [\delta_{e_3,e_3}] \rangle) = (1)$.

3.5. **1-dimensional annihilator extensions of $\mathcal{E}_{3,4}$.** As $\mathcal{H}(\mathcal{E}_{3,4} \times \mathcal{E}_{3,4}, \mathbb{F})$ is spanned by $\langle [\delta_{e_3,e_3}] \rangle$, we may assume $\theta = \delta_{e_3,e_3}$. Therefore we get the algebra $\mathcal{E}_{4,9} : e_1^2 = e_2, e_2^2 = e_3, e_3^2 = e_4$.

3.6. **2-dimensional annihilator extensions of $\mathcal{E}_{2,1}$.** Here $\mathcal{H}(\mathcal{E}_{2,1} \times \mathcal{E}_{2,1}, \mathbb{F}) = \langle [\delta_{e_1,e_1}], [\delta_{e_2,e_2}] \rangle$ and therefore we get only one algebra, namely $\mathcal{E}_{4,9} : e_1^2 = e_3, e_2^2 = e_4$.

Theorem 3.2. *Any 4-dimensional nilpotent evolution algebra over an algebraically closed field $\mathbb{F}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{E}$	*Multiplication Table*	$\mathcal{B}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$	$\mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F})$	$ann(\mathcal{E})$
$\mathcal{E}_{4,1}$			$Sym(\mathcal{E}_{4,1})$	$\mathcal{E}_{4,1}$
$\mathcal{E}_{4,2}$	$e_1^2 = e_2$.	$\langle \delta_{e_1,e_1} \rangle$	$\langle [\delta_{e_2,e_2}], [\delta_{e_3,e_3}], [\delta_{e_4,e_4}] \rangle$	$\langle e_2, e_3, e_4 \rangle$
$\mathcal{E}_{4,3}$	$e_1^2 = e_3, e_2^2 = e_3$.	$\langle \delta_{e_1,e_1} + \delta_{e_2,e_2} \rangle$	$\langle [\delta_{e_1,e_1}], [\delta_{e_3,e_3}], [\delta_{e_4,e_4}] \rangle$	$\langle e_3, e_4 \rangle$
$\mathcal{E}_{4,4}$	$e_1^2 = e_2, e_2^2 = e_3$.	$\langle \delta_{e_1,e_1}, \delta_{e_2,e_2} \rangle$	$\langle [\delta_{e_3,e_3}], [\delta_{e_4,e_4}] \rangle$	$\langle e_3, e_4 \rangle$
$\mathcal{E}_{4,5}$	$e_1^2 = e_4, e_2^2 = e_4, e_3^2 = e_4$.	$\langle \delta_{e_1,e_1} + \delta_{e_2,e_2} + \delta_{e_3,e_3} \rangle$	$\langle [\delta_{e_2,e_2}], [\delta_{e_3,e_3}], [\delta_{e_4,e_4}] \rangle$	$\langle e_4 \rangle$
$\mathcal{E}_{4,6}$	$e_1^2 = e_2, e_2^2 = e_4, e_3^2 = e_4$.	$\langle \delta_{e_1,e_1}, \delta_{e_2,e_2} + \delta_{e_3,e_3} \rangle$	$\langle [\delta_{e_3,e_3}], [\delta_{e_4,e_4}] \rangle$	$\langle e_4 \rangle$
$\mathcal{E}_{4,7}$	$e_1^2 = e_3, e_2^2 = e_3, e_3^2 = e_4$.	$\langle \delta_{e_1,e_1} + \delta_{e_2,e_2}, \delta_{e_3,e_3} \rangle$	$\langle [\delta_{e_2,e_2}], [\delta_{e_4,e_4}] \rangle$	$\langle e_4 \rangle$
$\mathcal{E}_{4,8}$	$e_1^2 = e_3 + e_4, e_2^2 = e_3, e_3^2 = e_4$.	$\langle \delta_{e_1,e_1} + \delta_{e_2,e_2}, \delta_{e_1,e_1} + \delta_{e_3,e_3} \rangle$	$\langle [\delta_{e_3,e_3}], [\delta_{e_4,e_4}] \rangle$	$\langle e_4 \rangle$
$\mathcal{E}_{4,9}$	$e_1^2 = e_2, e_2^2 = e_3, e_3^2 = e_4$.	$\langle \delta_{e_1,e_1}, \delta_{e_2,e_2}, \delta_{e_3,e_3} \rangle$	$\langle [\delta_{e_4,e_4}] \rangle$	$\langle e_4 \rangle$
$\mathcal{E}_{4,10}$	$e_1^2 = e_3, e_2^2 = e_4$.	$\langle \delta_{e_1,e_1}, \delta_{e_2,e_2} \rangle$	$\langle [\delta_{e_3,e_3}], [\delta_{e_4,e_4}] \rangle$	$\langle e_3, e_4 \rangle$

TABLE 2. Nilpotent evolution algebras of dimension four.

4. Five-dimensional nilpotent evolution algebras

In this section we give a complete classification of five-dimension nilpotent evolution algebras over an algebraically closed field $\mathbb{F}$.

4.1. **Nilpotent evolution algebras with annihilator components.** By Definition 2.14, we get the algebras $\mathcal{E}_{5,i} = \mathcal{E}_{4,i} \oplus \mathcal{E}_{1,1}$, $i = 1, \ldots, 10$.

4.2. **1-dimensional annihilator extensions of** $\mathcal{E}_{4,1}$**.** Here $Aut(\mathcal{E}_{4,1}) = GL(\mathcal{E}_{4,1})$. Moreover, $\mathcal{H}(\mathcal{E} \times \mathcal{E}, \mathbb{F}) = Sym(\mathcal{E}_{4,1})$ and $ann(\mathcal{E}_{4,1}) = \mathcal{E}_{4,1}$. Let $\theta = \sum_{i=1}^{4} \alpha_i \delta_{e_i,e_i}$ with $\alpha_1\alpha_2\alpha_3\alpha_4 \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \alpha_2^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \alpha_3^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \alpha_4^{-\frac{1}{2}} \end{bmatrix}.$$

Then $\phi\theta = \sum_{i=1}^{4} \delta_{e_i,e_i}$. So we may assume that $\theta = \sum_{i=1}^{4} \delta_{e_i,e_i}$. Therefore we get the algebra

$$\mathcal{E}_{5,11} : e_1^2 = e_2^2 = e_3^2 = e_4^2 = e_5.$$

4.3. **1-dimensional annihilator extensions of** $\mathcal{E}_{4,2}$**.** The automorphism group $Aut(\mathcal{E}_{4,2})$ consists of

$$\phi = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11}^2 & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} : a_{11}^3 (a_{33}a_{44} - a_{34}a_{43}) \neq 0$$

Let $[\theta] = \alpha_1 [\delta_{e_2,e_2}] + \alpha_2 [\delta_{e_3,e_3}] + \alpha_3 [\delta_{e_4,e_4}]$ with $\alpha_1\alpha_2\alpha_3 \neq 0$. Consider the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{-\frac{1}{4}} & 0 & 0 & 0 \\ 0 & \alpha_1^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \alpha_2^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \alpha_3^{-\frac{1}{2}} \end{bmatrix}.$$

Then $[\phi\theta] = [\delta_{e_2,e_2}] + [\delta_{e_3,e_3}] + [\delta_{e_4,e_4}]$. So we get the algebra

$$\mathcal{E}_{5,12} : e_1^2 = e_2, e_2^2 = e_3^2 = e_4^2 = e_5.$$

4.4. **1-dimensional annihilator extensions of** $\mathcal{E}_{4,3}$**.** The automorphism group $Aut(\mathcal{E}_{4,3})$ consists of

$$\phi = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{bmatrix},$$

such that

$$\begin{aligned} a_{11}a_{12} + a_{22}a_{21} &= 0, \\ a_{11}^2 + a_{21}^2 - a_{33} &= 0, \\ a_{12}^2 + a_{22}^2 - a_{33} &= 0, \\ a_{33}a_{44}(a_{11}a_{22} - a_{12}a_{21}) &\neq 0. \end{aligned}$$

Let $[\theta] = \alpha_1 [\delta_{e_1,e_1}] + \alpha_2 [\delta_{e_3,e_3}] + \alpha_3 [\delta_{e_4,e_4}]$ with $\alpha_2\alpha_3 \neq 0$. Suppose first that $\alpha_1 = 0$. Consider the following automorphism

$$\phi = \begin{bmatrix} \alpha_2^{-\frac{1}{4}} & 0 & 0 & 0 \\ 0 & \alpha_2^{-\frac{1}{4}} & 0 & 0 \\ 0 & 0 & \alpha_2^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \alpha_3^{-\frac{1}{2}} \end{bmatrix}.$$

Then $[\phi\theta] = [\delta_{e_3,e_3}] + [\delta_{e_4,e_4}]$. So we get the algebra

$$\mathcal{E}_{5,13} : e_1^2 = e_2^2 = e_3, e_3^2 = e_4^2 = e_5.$$

Assume now that $\alpha_1 = 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{\frac{1}{2}}\alpha_2^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \alpha_1^{\frac{1}{2}}\alpha_2^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \alpha_1\alpha_2^{-1} & 0 \\ 0 & 0 & 0 & \alpha_1\alpha_2^{-\frac{1}{2}}\alpha_3^{-\frac{1}{2}} \end{bmatrix}.$$

Then $[\phi\theta] = \alpha_1^2\alpha_2^{-1}\left([\delta_{e_1,e_1}] + [\delta_{e_3,e_3}] + [\delta_{e_4,e_4}]\right)$. So we may assume that $\theta = \delta_{e_1,e_1} + \delta_{e_3,e_3} + \delta_{e_4,e_4}$. Hence we get the algebra

$$\mathcal{E}_{5,14} : e_1^2 = e_3 + e_5, e_2^2 = e_3, e_3^2 = e_4^2 = e_5.$$

Further, $\mathcal{E}_{5,13}, \mathcal{E}_{5,14}$ are non-isomorphic since $\Psi\left(\langle [\delta_{e_3,e_3}] + [\delta_{e_4,e_4}]\rangle\right) = (2)$ while $\Psi\left(\langle [\delta_{e_1,e_1}] + [\delta_{e_3,e_3}] + [\delta_{e_4,e_4}]\rangle\right) = (1)$.

4.5. **1-dimensional annihilator extensions of $\mathcal{E}_{4,4}$.** The automorphism group $Aut\left(\mathcal{E}_{4,4}\right)$ consists of

$$\phi = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11}^2 & 0 & 0 \\ a_{31} & 0 & a_{11}^4 & 0 \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix} : a_{11}a_{44} \neq 0.$$

Let $[\theta] = \alpha_1\left[\delta_{e_3,e_3}\right] + \alpha_2\left[\delta_{e_4,e_4}\right]$ with $\alpha_1\alpha_2 \neq 0$. Consider the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{-\frac{1}{8}} & 0 & 0 & 0 \\ 0 & \alpha_1^{-\frac{1}{4}} & 0 & 0 \\ 0 & 0 & \alpha_1^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \alpha_2^{-\frac{1}{2}} \end{bmatrix}.$$

Then $[\phi\theta] = [\delta_{e_3,e_3}] + [\delta_{e_4,e_4}]$. Hence we get the algebra

$$\mathcal{E}_{5,15} : e_1^2 = e_2, e_2^2 = e_3, e_3^2 = e_4^2 = e_5.$$

4.6. **1-dimensional annihilator extensions of $\mathcal{E}_{4,5}$.** The automorphism group $Aut\left(\mathcal{E}_{4,5}\right)$ consists of

$$\phi = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

such that

$$\begin{aligned} a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= 0, \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0, \\ a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0, \\ a_{11}^2 + a_{21}^2 + a_{31}^2 - a_{44} &= 0, \\ a_{12}^2 + a_{22}^2 + a_{32}^2 - a_{44} &= 0, \\ a_{13}^2 + a_{23}^2 + a_{33}^2 - a_{44} &= 0 \\ \det\phi &\neq 0. \end{aligned}$$

Let $[\theta] = \alpha_1\left[\delta_{e_2,e_2}\right] + \alpha_2\left[\delta_{e_3,e_3}\right] + \alpha_3\left[\delta_{e_4,e_4}\right]$ with $\alpha_3 \neq 0$. Let us consider the following cases:

(1) $\alpha_1 = \alpha_2 = 0$. Then we may assume that $\theta = \delta_{e_4,e_4}$. So we get the algebra

$$\mathcal{E}_{5,16} : e_1^2 = e_2^2 = e_3^2 = e_4, e_4^2 = e_5.$$

(2) $\alpha_1 \neq 0, \alpha_2 = 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \alpha_1^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \alpha_1^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \alpha_1\alpha_3^{-1} \end{bmatrix}.$$

Then $[\phi\theta] = \alpha_1^2\alpha_3^{-1}\left([\delta_{e_2,e_2}] + [\delta_{e_4,e_4}]\right)$. So we may assume that $\theta = \delta_{e_2,e_2} + \delta_{e_4,e_4}$. Therefore we get the algebra

$$\mathcal{E}_{5,17} : e_1^2 = e_3^2 = e_4, e_2^2 = e_4 + e_5, e_4^2 = e_5.$$

(3) $\alpha_1 = 0, \alpha_2 \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} \alpha_2^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & \alpha_2^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 \\ 0 & \alpha_2^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & \alpha_2\alpha_3^{-1} \end{bmatrix}.$$

Then $[\phi\theta] = \alpha_2^2\alpha_3^{-1}\left([\delta_{e_2,e_2}] + [\delta_{e_4,e_4}]\right)$. So, we again get the algebra $\mathcal{E}_{5,17}$.

(4) $\alpha_1 = \alpha_2, \alpha_1^2 \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} 0 & \sqrt{-\alpha_1\alpha_3^{-1}} & 0 & 0 \\ 0 & 0 & \sqrt{-\alpha_1\alpha_3^{-1}} & 0 \\ \sqrt{-\alpha_1\alpha_3^{-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_1\alpha_3^{-1} \end{bmatrix}.$$

Then $[\phi\theta] = -\alpha_1^2\alpha_3^{-1}\left[\delta_{e_1,e_1} + \delta_{e_3,e_3}\right] + \alpha_1^2\alpha_3^{-1}\left[\delta_{e_4,e_4}\right] = \alpha_1^2\alpha_3^{-1}\left([\delta_{e_2,e_2}] + [\delta_{e_4,e_4}]\right)$. So we may assume that $\theta = \delta_{e_2,e_2} + \delta_{e_4,e_4}$. So, we again get the algebra $\mathcal{E}_{5,17}$.

(5) $\alpha_1\alpha_2\left(\alpha_1 - \alpha_2\right) \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \alpha_1^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \alpha_1^{\frac{1}{2}}\alpha_3^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \alpha_1\alpha_3^{-1} \end{bmatrix}.$$

Then $[\phi\theta] = \alpha_1^2\alpha_3^{-1}\left([\delta_{e_2,e_2}] + \alpha_1^{-1}\alpha_2\left[\delta_{e_3,e_3}\right] + [\delta_{e_4,e_4}]\right)$. So we may assume that $\theta = \delta_{e_2,e_2} + \alpha_1^{-1}\alpha_2\delta_{e_3,e_3} + \delta_{e_4,e_4}$. Set $\alpha = \alpha_1^{-1}\alpha_2$. Then $\alpha \neq 0, 1$. So we get the algebras

$$\mathcal{E}_{5,18}^{\alpha\in\mathbb{F}^*-\{1\}} : e_1^2 = e_4, e_2^2 = e_4 + e_5, e_3^2 = e_4 + \alpha e_5, e_4^2 = e_5.$$

Since $\Psi\left(\langle[\delta_{e_2,e_2}] + [\delta_{e_4,e_4}]\rangle\right) = (2)$ and $\Psi\left(\langle[\delta_{e_2,e_2}] + \alpha\left[\delta_{e_3,e_3}\right] + [\delta_{e_4,e_4}]\rangle\right) = (1)$, $\mathcal{E}_{5,17}$ is non-isomorphic to $\mathcal{E}_{5,18}^{\alpha}$. Further, we claim that $\mathcal{E}_{5,18}^{\alpha}$ is isomorphic to $\mathcal{E}_{5,18}^{\beta}$ if and only if $\beta = \alpha, \alpha^{-1}, 1-\alpha, (1-\alpha)^{-1}, \alpha\left(\alpha-1\right)^{-1}, \alpha^{-1}\left(\alpha-1\right)$. To see this, suppose first that $\mathcal{E}_{5,18}^{\alpha} \cong \mathcal{E}_{5,18}^{\beta}$. Then there exists a $\phi \in Aut\left(\mathcal{E}_{4,5}\right)$ such that $\phi\left([\delta_{e_2,e_2}] + \alpha\left[\delta_{e_3,e_3}\right] + [\delta_{e_4,e_4}]\right) = \lambda\left([\delta_{e_2,e_2}] + \beta\left[\delta_{e_3,e_3}\right] + [\delta_{e_4,e_4}]\right)$ for some $\lambda \in \mathbb{F}^*$. This the amounts to the following polynomial equations:

$$\begin{aligned} a_{22}^2 + \alpha a_{32}^2 + a_{42}^2 - \left(a_{21}^2 + \alpha a_{31}^2 + a_{41}^2\right) &= \lambda, \\ a_{23}^2 + \alpha a_{33}^2 + a_{43}^2 - \left(a_{21}^2 + \alpha a_{31}^2 + a_{41}^2\right) &= \lambda\beta, \\ a_{44}^2 &= \lambda, \\ a_{21}a_{22} + a_{41}a_{42} + \alpha a_{31}a_{32} &= 0, \\ a_{21}a_{23} + a_{41}a_{43} + \alpha a_{31}a_{33} &= 0, \\ a_{22}a_{23} + a_{42}a_{43} + \alpha a_{32}a_{33} &= 0, \\ a_{41}a_{44} &= 0, \\ a_{42}a_{44} &= 0, \\ a_{43}a_{44} &= 0, \end{aligned}$$

In addition to these equations we have

$$\begin{aligned} a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= 0, \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0, \\ a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0, \\ a_{11}^2 + a_{21}^2 + a_{31}^2 - a_{44} &= 0, \\ a_{12}^2 + a_{22}^2 + a_{32}^2 - a_{44} &= 0, \\ a_{13}^2 + a_{23}^2 + a_{33}^2 - a_{44} &= 0, \\ \det\phi &\neq 0, \end{aligned}$$

which ensure that $\phi \in Aut\left(\mathcal{E}_{4,5}\right)$. To solve these equations, it is convenient to compute the Grőbner basis of the ideal generated by all of the previous polynomial equations to get an equivalent set of equations which may be easier to solve. This way, we get (among others) the following equation:

$\alpha^6\beta^4-2\alpha^6\beta^3+\alpha^6\beta^2-3\alpha^5\beta^4+6\alpha^5\beta^3-3\alpha^5\beta^2-\alpha^4\beta^6+3\alpha^4\beta^5-5\alpha^4\beta^3+3\alpha^4\beta-\alpha^4+2\alpha^3\beta^6-6\alpha^3\beta^5+5\alpha^3\beta^4+5\alpha^3\beta^2-6\alpha^3\beta+2\alpha^3-\alpha^2\beta^6+3\alpha^2\beta^5-5\alpha^2\beta^3+3\alpha^2\beta-\alpha^2-3\alpha\beta^4+6\alpha\beta^3-3\alpha\beta^2+\beta^4-2\beta^3+\beta^2=0$. This is equivalent to $((1-\alpha)\beta-1)(\beta-(1-\alpha))(\alpha\beta-1)(\beta-\alpha)(\alpha\beta-(\alpha-1))(\beta(\alpha-1)-\alpha)=0$, as claimed. Conversely, let $\beta=\alpha,\alpha^{-1},1-\alpha,(1-\alpha)^{-1},\alpha(\alpha-1)^{-1},\alpha^{-1}(\alpha-1)$. Consider the following automorphisms

$$\phi_1 = \begin{bmatrix} -\alpha^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & \alpha^{\frac{1}{2}} & 0 \\ 0 & \alpha^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \phi_2 = \begin{bmatrix} 0 & \sqrt{-1} & 0 & 0 \\ \sqrt{-1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \phi_3 = \begin{bmatrix} 0 & 0 & (\alpha-1)^{\frac{1}{2}} & 0 \\ \frac{(\alpha-1)^3}{\sqrt{(\alpha-1)^5}} & 0 & 0 & 0 \\ 0 & (\alpha-1)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & \alpha-1 \end{bmatrix},$$

$$\phi_4 = \begin{bmatrix} 0 & 0 & (1-\alpha)^{\frac{1}{2}} & 0 \\ 0 & (1-\alpha)^{\frac{1}{2}} & 0 & 0 \\ \frac{(\alpha-1)^3}{\sqrt{-(\alpha-1)^5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha-1 \end{bmatrix}, \phi_5 = \begin{bmatrix} 0 & \sqrt{-\alpha} & 0 & 0 \\ 0 & 0 & \sqrt{-\alpha} & 0 \\ -\sqrt{-\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Then

$$\begin{aligned}
\phi_1([\delta_{e_2,e_2}]+\alpha[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]) &= \alpha^2\left([\delta_{e_2,e_2}]+\alpha^{-1}[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]\right),\\
\phi_2([\delta_{e_2,e_2}]+\alpha[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]) &= [\delta_{e_2,e_2}]+(1-\alpha)[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}],\\
\phi_3([\delta_{e_2,e_2}]+\alpha[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]) &= (\alpha-1)^2\left([\delta_{e_2,e_2}]+(1-\alpha)^{-1}[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]\right),\\
\phi_4([\delta_{e_2,e_2}]+\alpha[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]) &= (\alpha-1)^2\left([\delta_{e_2,e_2}]+\alpha(\alpha-1)^{-1}[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]\right),\\
\phi_5([\delta_{e_2,e_2}]+\alpha[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]) &= \alpha^2\left([\delta_{e_2,e_2}]+\alpha^{-1}(\alpha-1)[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]\right).
\end{aligned}$$

Therefore, $\mathcal{E}_{5,18}^{\alpha}\cong\mathcal{E}_{5,18}^{\alpha^{-1}}\cong\mathcal{E}_{5,18}^{1-\alpha}\cong\mathcal{E}_{5,18}^{(1-\alpha)^{-1}}\cong\mathcal{E}_{5,18}^{\alpha(\alpha-1)^{-1}}\cong\mathcal{E}_{5,18}^{\alpha^{-1}(\alpha-1)}$. This completes the proof of the claim.

4.7. **1-dimensional annihilator extensions of $\mathcal{E}_{4,6}$.** The automorphism group $Aut(\mathcal{E}_{4,6})$ consists of

$$\phi = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11}^2 & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & 0 & a_{43} & a_{11}^4 \end{bmatrix} : a_{33}^2 = a_{11}^4, a_{11} \neq 0.$$

Let $[\theta]=\alpha_1[\delta_{e_3,e_3}]+\alpha_2[\delta_{e_4,e_4}]$ with $\alpha_2\neq 0$. If $\alpha_1=0$, we then may assume that $\theta=\delta_{e_4,e_4}$. So we get the algebra

$$\mathcal{E}_{5,19}: e_1^2=e_2, e_2^2=e_3^2=e_4, e_4^2=e_5.$$

Otherwise, we choose ϕ to be the following automorphism

$$\phi = \begin{bmatrix} \sqrt[4]{\alpha_1\alpha_2^{-1}} & 0 & 0 & 0 \\ 0 & \sqrt{\alpha_1\alpha_2^{-1}} & 0 & 0 \\ 0 & 0 & \sqrt{\alpha_1\alpha_2^{-1}} & 0 \\ 0 & 0 & 0 & \alpha_1\alpha_2^{-1} \end{bmatrix}.$$

Then $[\phi\theta]=\alpha_1^2\alpha_2^{-1}([\delta_{e_3,e_3}]+[\delta_{e_4,e_4}])$. So we may assume that $\theta=\delta_{e_3,e_3}+\delta_{e_4,e_4}$. Hence we get the algebra

$$\mathcal{E}_{5,20}: e_1^2=e_2, e_2^2=e_4, e_3^2=e_4+e_5, e_4^2=e_5.$$

Since $\Psi(\langle[\delta_{e_4,e_4}]\rangle)=(3)$ and $\Psi(\langle[\delta_{e_3,e_3}]+[\delta_{e_4,e_4}]\rangle)=(2)$, $\mathcal{E}_{5,19}$ is non-isomorphic to $\mathcal{E}_{5,20}$.

4.8. **1-dimensional annihilator extensions of $\mathcal{E}_{4,7}$.** The automorphism group $Aut(\mathcal{E}_{4,7})$ consists of

$$\phi = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & a_{42} & 0 & a_{33}^2 \end{bmatrix},$$

such that

$$\begin{aligned} a_{11}a_{12}+a_{22}a_{21} &= 0,\\ a_{11}^2+a_{21}^2-a_{33} &= 0,\\ a_{12}^2+a_{22}^2-a_{33} &= 0,\\ a_{33}\left(a_{11}a_{22}-a_{12}a_{21}\right) &\neq 0. \end{aligned}$$

Let $[\theta]=\alpha_1\left[\delta_{e_2,e_2}\right]+\alpha_2\left[\delta_{e_4,e_4}\right]$ with $\alpha_2\neq 0$. If $\alpha_1=0$, we then may assume that $\theta=\delta_{e_4,e_4}$. So we get the algebra

$$\mathcal{E}_{5,21}: e_1^2=e_2^2=e_3, e_3^2=e_4, e_4^2=e_5.$$

Otherwise, we choose ϕ to be the following automorphism

$$\phi=\begin{bmatrix} \alpha_1^{\frac{1}{6}}\alpha_2^{-\frac{1}{6}} & 0 & 0 & 0\\ 0 & \alpha_1^{\frac{1}{6}}\alpha_2^{-\frac{1}{6}} & 0 & 0\\ 0 & 0 & \alpha_1^{\frac{1}{3}}\alpha_2^{-\frac{1}{3}} & 0\\ 0 & 0 & 0 & \alpha_1^{\frac{2}{3}}\alpha_2^{-\frac{2}{3}} \end{bmatrix}.$$

Then $[\phi\theta]=\alpha_1^{\frac{4}{3}}\alpha_2^{-\frac{1}{3}}\left(\left[\delta_{e_2,e_2}\right]+\left[\delta_{e_4,e_4}\right]\right)$. So we may assume that $\theta=\delta_{e_2,e_2}+\delta_{e_4,e_4}$. Hence we get the algebra

$$\mathcal{E}_{5,22}: e_1^2=e_3, e_2^2=e_3+e_5, e_3^2=e_4, e_4^2=e_5.$$

Further, $\mathcal{E}_{5,21}$, $\mathcal{E}_{5,22}$ are non-isomorphic since $\Psi\left(\left\langle\left[\delta_{e_4,e_4}\right]\right\rangle\right)=(3)$ while $\Psi\left(\left\langle\left[\delta_{e_2,e_2}\right]+\left[\delta_{e_4,e_4}\right]\right\rangle\right)=(2)$.

4.9. **1-dimensional annihilator extensions of** $\mathcal{E}_{4,8}$**.** The automorphism group $Aut\left(\mathcal{E}_{4,8}\right)$ consists of

$$\phi_1=\begin{bmatrix} a_{11} & 0 & 0 & 0\\ 0 & a_{22} & 0 & 0\\ 0 & 0 & a_{11}^2 & 0\\ a_{41} & a_{42} & 0 & a_{11}^2 \end{bmatrix}, \phi_2=\begin{bmatrix} 0 & a_{12} & 0 & 0\\ a_{21} & 0 & 0 & 0\\ 0 & 0 & a_{12}^2 & 0\\ a_{41} & a_{42} & -a_{12}^4 & a_{12}^4 \end{bmatrix}: a_{11}\neq 0, a_{11}^2=a_{22}^2, a_{12}\neq 0, a_{12}^2=a_{21}^2.$$

Let $[\theta]=\alpha_1\left[\delta_{e_3,e_3}\right]+\alpha_2\left[\delta_{e_4,e_4}\right]$ with $\alpha_2\neq 0$. Then we may assume that $\theta=\alpha\delta_{e_3,e_3}+\delta_{e_4,e_4}$ where $\alpha\in\mathbb{F}$. So we get the algebras

$$\mathcal{E}_{5,23}^{\alpha\in\mathbb{F}}: e_1^2=e_3+e_4, e_2^2=e_3+\alpha e_5, e_3^2=e_4, e_4^2=e_5.$$

Further, we claim that $\mathcal{E}_{5,23}^{\alpha}$ is isomorphic to $\mathcal{E}_{5,23}^{\beta}$ if and only if $\alpha=\beta$. To see this, let $\mathcal{E}_{5,23}^{\alpha}\cong\mathcal{E}_{5,23}^{\beta}$. Then there exist a $\phi\in Aut\left(\mathcal{E}_{4,8}\right)$ such that

$$\phi\left(\alpha\left[\delta_{e_3,e_3}\right]+\left[\delta_{e_4,e_4}\right]\right)=\lambda\left(\beta\left[\delta_{e_3,e_3}\right]+\left[\delta_{e_4,e_4}\right]\right);\lambda\in\mathbb{F}^*. \tag{4.1}$$

Suppose first that $\phi=\phi_1$. Then Eq. (4.1) amounts to the following polynomial equations:

$$\alpha a_{11}^4-a_{41}^2+a_{42}^2=\lambda\beta, a_{11}^4=\lambda, a_{41}a_{42}=0, a_{11}^2a_{41}=0, a_{11}^2a_{42}=0.$$

As $a_{11}\neq 0$ we have $a_{41}=a_{42}=0$. Consequently, we get $\alpha=\beta$. Assume now that $\phi=\phi_2$. Then Eq. (4.1) amounts to the following polynomial equations:

$$a_{12}^4\left(a_{12}^4+\alpha\right)-a_{41}^2+a_{42}^2=\lambda\beta, a_{12}^8=\lambda, a_{41}a_{42}=0, a_{12}^4a_{41}=0, a_{12}^4a_{41}=0, a_{12}^4a_{42}=0, a_{12}^4a_{42}=0, a_{12}^8=0.$$

The last equation implies that $a_{12}=0$, which is impossible as $\det\phi\neq 0$ (i.e., $\phi\notin\mathcal{S}_\theta\left(\mathcal{E}_{4,8}\right)$). Therefore, we have $\alpha=\beta$ when $\mathcal{E}_{5,23}^{\alpha}\cong\mathcal{E}_{5,23}^{\beta}$, as claimed.

4.10. **1-dimensional annihilator extensions of** $\mathcal{E}_{4,9}$**.** Since $\mathcal{H}\left(\mathcal{E}\times\mathcal{E},\mathbb{F}\right)=\left\langle\left[\delta_{e_4,e_4}\right]\right\rangle$, we therefore get the algebra

$$\mathcal{E}_{4,24}: e_1^2=e_2, e_2^2=e_3, e_3^2=e_4, e_4^2=e_5.$$

4.11. **1-dimensional annihilator extensions of** $\mathcal{E}_{4,10}$**.** The automorphism group $Aut\left(\mathcal{E}_{4,10}\right)$ consists of

$$\phi=\begin{bmatrix} a_{11} & 0 & 0 & 0\\ 0 & a_{22} & 0 & 0\\ a_{31} & 0 & a_{11}^2 & 0\\ a_{41} & a_{42} & 0 & a_{22}^2 \end{bmatrix}: a_{11}a_{22}\neq 0.$$

Let $[\theta] = \alpha_1 [\delta_{e_3,e_3}] + \alpha_2 [\delta_{e_4,e_4}]$ with $\alpha_1\alpha_2 \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} \alpha_1^{-\frac{1}{4}} & 0 & 0 & 0 \\ 0 & \alpha_2^{-\frac{1}{4}} & 0 & 0 \\ 0 & 0 & \alpha_1^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \alpha_2^{-\frac{1}{2}} \end{bmatrix}.$$

Then $[\phi\theta] = [\delta_{e_3,e_3}] + [\delta_{e_4,e_4}]$. So we may assume that $\theta = \delta_{e_3,e_3} + \delta_{e_4,e_4}$. Thus we get the algebra

$$\mathcal{E}_{4,25} : e_1^2 = e_3, e_2^2 = e_4, e_3^2 = e_4^2 = e_5.$$

4.12. **2-dimensional annihilator extensions of $\mathcal{E}_{3,1}$.** Here $Aut(\mathcal{E}_{4,1}) = GL(\mathcal{E}_{3,1})$ and $\mathcal{H}(\mathcal{E}\times\mathcal{E},\mathbb{F}) = Sym(\mathcal{E}_{3,1})$. Let $\theta_1, \theta_2 \in Sym(\mathcal{E}_{3,1})$ such that $\theta_1^{\perp} \cap \theta_2^{\perp} = 0$. Without loss of generality we may assume that $\theta_1 = \delta_{e_1,e_1} + \alpha\delta_{e_3,e_3}, \theta_2 = \delta_{e_2,e_2} + \beta\delta_{e_3,e_3}$. Therefore, $\Psi(\langle\theta_1,\theta_2\rangle) \in \{(2,1),(1,1)\}$. Suppose first that $\Psi(\langle\theta_1,\theta_2\rangle) = (2,1)$. Then we may assume that $\theta_1 = \delta_{e_1,e_1}, \theta_2 = \delta_{e_2,e_2} + \beta\delta_{e_3,e_3}$, where $\beta \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta^{-\frac{1}{2}} \end{bmatrix}.$$

Then $\phi\theta_1 = \theta_1, \phi\theta_2 = \delta_{e_2,e_2} + \delta_{e_3,e_3}$. So we get the algebra

$$\mathcal{E}_{4,26} : e_1^2 = e_4, e_2^2 = e_3^2 = e_5.$$

Assume now that $\Psi(\langle\theta_1,\theta_2\rangle) = (1,1)$. Then $\alpha\beta \neq 0$. Let ϕ be the following automorphism

$$\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha^{-\frac{1}{2}}\beta^{\frac{1}{2}} & 0 \\ 0 & 0 & \alpha^{-\frac{1}{2}} \end{bmatrix}.$$

Then $\phi\theta_1 = \delta_{e_1,e_1} + \delta_{e_3,e_3}, \phi\theta_2 = \alpha^{-1}\beta(\delta_{e_2,e_2} + \delta_{e_3,e_3})$. So we may assume that $\theta_1 = \delta_{e_1,e_1} + \delta_{e_3,e_3}, \theta_2 = \delta_{e_2,e_2} + \delta_{e_3,e_3}$. Hence we get the algebra

$$\mathcal{E}_{4,27} : e_1^2 = e_4, e_2^2 = e_3^2 = e_4 + e_5.$$

4.13. **2-dimensional annihilator extensions of $\mathcal{E}_{3,2}$.** As $\mathcal{H}(\mathcal{E}_{3,2}\times\mathcal{E}_{3,2},\mathbb{F}) = \langle[\delta_{e_2,e_2}],[\delta_{e_3,e_3}]\rangle$ we get only one algebra, namely

$$\mathcal{E}_{4,28} : e_1^2 = e_2, e_2^2 = e_4, e_3^2 = e_5.$$

4.14. **2-dimensional annihilator extensions of $\mathcal{E}_{3,3}$.** As $\mathcal{H}(\mathcal{E}_{3,3}\times\mathcal{E}_{3,3},\mathbb{F}) = \langle[\delta_{e_2,e_2}],[\delta_{e_3,e_3}]\rangle$ we get only one algebra, namely

$$\mathcal{E}_{4,29} : e_1^2 = e_3, e_2^2 = e_3 + e_4, e_3^2 = e_5.$$

Theorem 4.1. *Any five-dimensional nilpotent evolution algebra with a natural basis $\{e_1,\dots e_5\}$ over an algebraically closed field $\mathbb{F}$ is isomorphic to one of the following algebras:*

- $\mathcal{E}_{5,1}$: *all products are zero.*
- $\mathcal{E}_{5,2} : e_1^2 = e_2.$
- $\mathcal{E}_{5,3} : e_1^2 = e_3, e_2^2 = e_3.$
- $\mathcal{E}_{5,4} : e_1^2 = e_2, e_2^2 = e_3.$
- $\mathcal{E}_{5,5} : e_1^2 = e_4, e_2^2 = e_4, e_3^2 = e_4.$
- $\mathcal{E}_{5,6} : e_1^2 = e_2, e_2^2 = e_4, e_3^2 = e_4.$
- $\mathcal{E}_{5,7} : e_1^2 = e_3, e_2^2 = e_3, e_3^2 = e_4.$
- $\mathcal{E}_{5,8} : e_1^2 = e_3 + e_4, e_2^2 = e_3, e_3^2 = e_4.$
- $\mathcal{E}_{5,9} : e_1^2 = e_2, e_2^2 = e_3, e_3^2 = e_4.$
- $\mathcal{E}_{5,10} : e_1^2 = e_3, e_2^2 = e_4.$
- $\mathcal{E}_{5,11} : e_1^2 = e_2^2 = e_3^2 = e_4^2 = e_5.$
- $\mathcal{E}_{5,12} : e_1^2 = e_2, e_2^2 = e_3^2 = e_4^2 = e_5.$
- $\mathcal{E}_{5,13} : e_1^2 = e_2^2 = e_3, e_3^2 = e_4^2 = e_5.$
- $\mathcal{E}_{5,14} : e_1^2 = e_3 + e_5, e_2^2 = e_3, e_3^2 = e_4^2 = e_5.$
- $\mathcal{E}_{5,15} : e_1^2 = e_2, e_2^2 = e_3, e_3^2 = e_4^2 = e_5.$
- $\mathcal{E}_{5,16} : e_1^2 = e_2^2 = e_3^2 = e_4, e_4^2 = e_5.$
- $\mathcal{E}_{5,17} : e_1^2 = e_3^2 = e_4, e_2^2 = e_4 + e_5, e_4^2 = e_5.$
- $\mathcal{E}_{5,18}^{\alpha\in\mathbb{F}^*-\{1\}} : e_1^2 = e_4, e_2^2 = e_4 + e_5, e_3^2 = e_4 + \alpha e_5, e_4^2 = e_5.$

- $\mathcal{E}_{5,19}: e_1^2 = e_2, e_2^2 = e_3^2 = e_4, e_4^2 = e_5$.
- $\mathcal{E}_{5,20}: e_1^2 = e_2, e_2^2 = e_4, e_3^2 = e_4 + e_5, e_4^2 = e_5$.
- $\mathcal{E}_{5,21}: e_1^2 = e_2^2 = e_3, e_3^2 = e_4, e_4^2 = e_5$.
- $\mathcal{E}_{5,22}: e_1^2 = e_3, e_2^2 = e_3 + e_5, e_3^2 = e_4, e_4^2 = e_5$.
- $\mathcal{E}_{5,23}^{\alpha\in\mathbb{F}}: e_1^2 = e_3 + e_4, e_2^2 = e_3 + \alpha e_5, e_3^2 = e_4, e_4^2 = e_5$.
- $\mathcal{E}_{4,24}: e_1^2 = e_2, e_2^2 = e_3, e_3^2 = e_4, e_4^2 = e_5$.
- $\mathcal{E}_{4,25}: e_1^2 = e_3, e_2^2 = e_4, e_3^2 = e_4^2 = e_5$.
- $\mathcal{E}_{4,26}: e_1^2 = e_4, e_2^2 = e_3^2 = e_5$.
- $\mathcal{E}_{4,27}: e_1^2 = e_4, e_2^2 = e_3^2 = e_4 + e_5$.
- $\mathcal{E}_{4,28}: e_1^2 = e_2, e_2^2 = e_4, e_3^2 = e_5$.
- $\mathcal{E}_{4,29}: e_1^2 = e_3, e_2^2 = e_3 + e_4, e_3^2 = e_5$.

Among these algebras there is precisely the following isomorphism:

- $\mathcal{E}_{5,18}^{\alpha} \cong \mathcal{E}_{5,18}^{\beta}$ *if and only if* $(\beta-\alpha)\left(\beta-\frac{1}{\alpha}\right)(\beta-1+\alpha)\left(\beta-\frac{1}{1-\alpha}\right)\left(\beta-\frac{\alpha-1}{\alpha}\right)\left(\beta-\frac{\alpha}{\alpha-1}\right)=0$.

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